Mathematics 222B Lecture 18 Notes

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1 Definition of Hyperbolicity

1.1 Working definition of hyperbolicity

Let's return to our general discussion of hyperbolicity. We initially gave a working definition of hyperbolicity:

- Order of t derivatives = order of x-derivatives
- (Local) well-posedness of the initial value problem.

The purpose of the first condition is to ensure that we have a **finite speed of propagation**. This is as opposed to some other equations, where you may have compactly supported initial data, but immediately after $t = 0$, the solution is no longer compactly supported. The finite speed of propagation is very related to Lorentzian geometry.

We would also like to have an algebraic definition of hyperbolicity. Here, we will give the standard definition you may see in a paper or textbook. We gave the working definition first because there are some hyperbolic PDEs which are badly behaved (e.g. you can't prove local well-posedness of the initial value problem without extra assumptions). In what follows, compare with the PDE $P\phi = f$, where

$$P\phi = -\partial_t^2 \phi + \partial_j(a^{k,j}\partial_k \phi) + b^j \partial_j \phi + c\phi.$$

This is a special case of hyperbolicity for second order linear PDEs.

1.2 Hyperbolicity for first-order systems

The solution we want will take the form of $\Phi : \mathbb{R}_t \times \mathbb{R}^d \to \mathbb{R}^n$, with the equation

$$\partial_t \Phi^J + (B^j)^J_K \partial_j \Phi^K.$$

We can express this in matrix notation as

$$\partial_t \Phi + B^j \partial_j \Phi = F.$$

Here, $B^j = (B^j)^J_K$ is an $n \times n$ matrix valued function on $\mathbb{R}_t \times \mathbb{R}^d$, and $F : \mathbb{R}_t \times \mathbb{R}^d \to \mathbb{R}^n$. The initial condition is $\Phi|_{t=0} = \Phi_0$.

Definition 1.1. Let the **symbol** of $B^j \partial_j$ be $\sigma(t, x; \xi) = \xi_j B^j(t, x)$, where $\xi \in \mathbb{R}^d$. We say that a first-order equation is **hyperbolic** if $\sigma(t, x; \xi)$ has n real eigenvalues for each t, x, ξ.

Theorem 1.1 (Constant coefficient case)**.** *Assume B^j is independent of (t, x). Then hyperbolicity implies local well-posedness of the initial value problem.*

Proof. The proof involves Fourier analysis, and you can find it in section 7.3 of Evans' textbook. □

If B^j is constant and σ has a non-real eigenvalue, then there exists a **plane wave solution**

$$\Phi = Ae^{i(x \cdot \xi + t\omega)},$$

where $\operatorname{Im} \omega \neq 0$. If $\operatorname{Im} \omega < 0$, this solution experiences exponential growth in time. This can be formalized into an ill-posedness statement. This should motivate our definition of hyperbolicity.

However, in the variable coefficient case, we need stronger conditions to ensure local well-posedness.

Definition 1.2. A first-order equation is **symmetric hyperbolic** if each $B^j(t, x)$ is symmetric for all t, x. If there exists a similarity transformation $P(t, x)$ sending $\Phi \mapsto \widetilde{\Phi} = P\Phi$ such that the transformed equation is symmetric hyperbolic, then the equation is called **symmetrizable hyperbolic**.

Theorem 1.2. *A symmetric hyperbolic first-order equation (with regularity assumptions on B) has local well-posedness of the initial value problem.*

Proof. This proof is by the energy method. You can find a proof in 7.3 in Evans' textbook, but the method we have presented in class is closer to the presentation in Chapter 7 of Ringström's book. □

Definition 1.3. A hyperbolic first-order system is said to be **strictly hyperbolic** if all n real eigenvalues are distinct (for all t, x, ξ)

$$\lambda_1(t, x; \xi) < \cdots < \lambda_n(t, x, ; \xi).$$

This is a useful definition when the spatial dimension is $d = 1$. In this case, these eigenvalue separation conditions help us use the method of characteristics to solve this system (normally you can only solve scalar equations in this way). This is not discussed in Evans' book, but it is discussed in *Hyperbolic Conservation Laws* by Dafemos.

1.3 Hyperbolicity for second-order, linear, scalar PDEs

Here, we give a notion of hyperbolicity that generalizes our wave equation $\Box\phi = 0$. We have

$$P\phi = \partial_\mu(g^{\mu,\nu}\partial_\nu\phi) + b^\mu\partial_\mu\phi + c\phi.$$

Let's focus on $g^{\mu,\nu}$, the important part. This brings us to the idea of Lorentzian (inverse) metrics.

Let $(g^{-1})^{\mu,\nu}(t,x)$ be a symmetric $(1+d)\times(1+d)$ matrix with signature $(-,+,+,\dots,+)$. (Compare this to if the case where $M = \operatorname{diag}(-1,1,1,\dots,1)$, so the wave equation can be written as $\partial_\mu(m^{\mu,\nu}\partial_\nu\phi) = 0$.) Let $g = g_{\mu,\nu}$.

Definition 1.4. A **Lorentzian manifold** is a pair $(\mathcal{M}, g)$, where $\mathcal{M}$ is a $(1+d)$-dimensional smooth manifold, and g is a symmetric, covariant 2-tensor with signature $(-,+,+,\dots,+)$.

The key difference from Riemannian geometry is due to the following.

Lemma 1.1. *Let $Q(\xi)$ be a quadratic form $q^{\alpha,\beta}\xi_\alpha\xi_\beta$. If q has no zero eigenvalue and has at least one negative and positive eigenvalues, then $\{\xi : Q(\xi) = 0\}$ determines q up to multiplication by a constant.*

The condition $g^{\alpha,\beta}\xi_\alpha\xi_\beta = 0$ determines $g^{\alpha,\beta}$ (up to a constant). This explains why Lorentzian geometry is a natural setting for Einstein's equations. If we have $g_{\alpha,\beta}v^\alpha v^\beta = 0$, the the zero set of P looks like a cone. This lemma tells you that these distinguished directions determine the behavior. In relativity, there are distinguished speeds, such as the speed of light. The tangent space at each point is made of velocity vectors. One way to think of this is that at each point, you get a cone, but you need some way to stitch these together; the Lorentzian metric is a natural way to do this.

Here is an algebraic lemma which connects the restricted class of PDEs that we have considered to this Lorentzian setting.

Lemma 1.2. *Let g be a Lorentzian metric, and let $p \in \mathcal{M}$. There exists a neighborhood $U \ni p$ and local coordinates $(x^o, \dots, x^d)$ in U such that $(g^{-1})^{0,j} = 0$ for all $j = 1, \dots, d$ and $(g^{-1})^{0,0} < -c$ for some $c > 0$. We may also ensure that $(g^{-1})^{j,k}\xi_j\xi_k \geq c_0|\xi|^2$ for some $c_0 > 0$.*

Corollary 1.1. *Locally,*

$$P\phi = \partial_\mu(g^{\mu,\nu}\partial_\nu\phi) + b^\mu\partial_\mu\phi + c\phi$$

can be put in the restricted form discussed earlier ($g^{0,j} = 0$, $g^{0,0} = -1$, and $g^{j,k}\xi_j\xi_k \geq \lambda|\xi|^2$). The condition $g^{0,0} = -1$ can be ensured by normalization at the level of the PDE.

Proof. Take x^0 such that $(g^{-1})^{\mu,\nu}(dx^0)_\mu(dx^0)_\nu < 0$; we say that such a dx^0 is **time-like**. We are looking for hypersurfaces that are transversal to the zero cones at each point. Take any local coordinates x^j near p in $\{x^0 = 0\}$. We want to transport x^j to other level surfaces of x^0 so that $(g^{-1})^{0,j} = 0$. Here is the procedure. Take a 1-form $(dx^0)_\mu$ and form a vector field $(g^{-1})^{\mu,\nu}(dx^0)_\nu = \nabla x^0$. If we write this in coordinates, this vector field is $(g^{-1})^{\mu,\nu}$.

We want to make sure that $(g^{-1})^{0,\mu}\partial_\mu x^j = 0$. So we construct this vector field ∇x^0 and then flow along the vector field. □

1.4 Geometric formulation of local well-posedness of the initial value problem

Here is the geometric idea concerning the initial time we start our initial conditions at. Just as in Riemannian geometry, we can create a Levi-Civita connection, which leads to parallel transport.

Definition 1.5. A C^1 curve γ is a **geodesic** if $\nabla_{\dot\gamma}\dot\gamma = 0$.

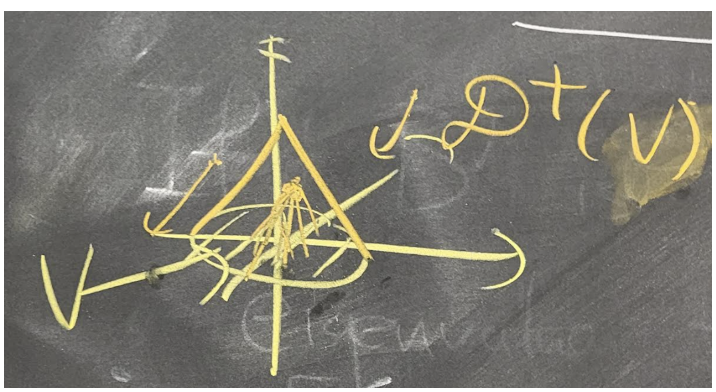
For a geodesic γ, $\frac{d}{dt}g(\dot\gamma,\dot\gamma) = 0$. The curve γ is called

$$\begin{cases} \text{timelike} & \text{if } g(\dot\gamma,\dot\gamma) < 0 \\ \text{null} & \text{if } g(\dot\gamma,\dot\gamma) < 0 \\ \text{spacelike} & \text{if } g(\dot\gamma,\dot\gamma) > 0. \end{cases}$$

Corollary 1.2. *If γ is a geodesic, there is a well-defined* ***causal*** *(i.e. timelike or null) character.*

Definition 1.6. A Lorentzian manifold $(\mathcal{M}, g)$ is **time-orientable** if there exists a non-vanishing vector field that is timelike everywhere.

Definition 1.7. Let $\mathcal{M}$ be a Lorentzian manifold, and let $V \subseteq \mathcal{M}$. The **causal future** of V is $\mathcal{J}^+(V) = \{\text{all } q \in \mathcal{M} \text{ with a future causal curve from } p \in V \text{ to } q\}$. We also let $\mathcal{D}^+(V) = \{q : J^-(q) : \text{ all causal past-pointing curves from } q \text{ meet } V\}$.



Here, future means the top half of the cone.

Definition 1.8. A **Cauchy hypersurface** is a spacelike hypersurface; i.e. a hyper surface with all tangent vectors spacelike

In this picture,

$$\mathcal{M} = \mathcal{D}^+(\Sigma) \cup \Sigma \cup \mathcal{D}^-(\Sigma).$$

Definition 1.9. Global hyperbolicity is when $(\mathcal{M}, g)$ is time orientable and there exists a Cauchy hypersurface.

Theorem 1.3. *Let $(\mathcal{M}, g)$ be globally hyperbolic with a Cauchy hypersurface Σ. Then*

$$\begin{cases} \Box_g \phi + B\phi + c\phi = f \\ (\phi, n_\Sigma \phi)|_\Sigma = (g, h) \end{cases}$$

is well-posed (existence and uniqueness). here. B is a vector field, c is a function, n_Σ is the unit normal to Σ, and

$$\Box_g \phi = \mathrm{div}_g(d\phi) = \frac{1}{\sqrt{|d+g|}} \partial_\mu ((g^{-1})^{\mu,\nu} \sqrt{|d+g|} \partial_\nu \phi).$$

The converse is also true.

What is interesting is there is a purely geometric formulation of this. A good reference for this story is Chapters 10 to 12 of Ringström's book.